A STRAIGHT LINE PLANE WITH PREASSIGNED CIRCLES

BY

EDGAR RAYMOND MULLINS, JR.

B.A., Grinnell College, 1947
A.M., University of Illinois, 1949

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS
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I. Introduction

An S. L. (straight line) space is a set termed "space," that is metric, finitely compact, internally convex, locally externally convex and such that any two distinct points are on at most one geodesic. It has been the object of intensive study by Busemann with respect to finite dimensional metric spaces that possess geodesics with the usual properties and so includes symmetric Finsler spaces. (see reference [1]).

The geodesics in any such plane are congruent to either the euclidean line or circle and so are called straight lines. Since both open and closed straight lines cannot exist in the same space, an S. L. space is designated open or closed accordingly. In two dimensions Busemann has shown that for the euclidean plane with a set of open Jordan curves such that any two distinct points of the plane are contained in exactly one curve of the system, an S. L. plane exists whose metric is topologically equivalent to the euclidean metric and for which the curves of the system are the straight lines. It is this result that enables us to conduct our study.

II. The Straight Lines

In the euclidean plane E_2 with the usual euclidean metric $e(\ )$, let there be given a system of curves $\varnothing(P)$, associated with every point P of E_2, with the following properties:

$[a_1]$ Each curve is simple and closed.

$[a_2]$ For any point $Q \neq P$ of E_2, there is exactly one curve of the system at P passing through Q.

$[a_3]$ For any real number $N > O$, there is at least one curve of the system of curves at P whose open interior contains the euclidean circle, center at P,

radius N, and at least one curve of the system contained in the open interior of the euclidean circle.

[a_4] If $B \neq P$ is an interior point of the closed interior of any curve of $\varnothing(P)$, then the closed interior of the curve through B in $\varnothing(P)$ is wholly contained in the first set.

The curve of $\varnothing(P)$ through A will be designated $c(P,A)$ and its closed interior $d(P,A)$. The closed exterior of $c(P,A)$ will be designated $\overline{d}(P,A)$. $d(P,P)$ will be the point P. The euclidean circle at P of radius r will be denoted by $k(P,r)$.

Since the curves of $\varnothing(P)$ are simple and closed, with each curve there is associated a real number, the minimum euclidean distance from P to the curve. This number can then be used as a parameter for $\varnothing(P)$. $\varnothing(P)$ is well ordered.

From this point, these curves associated with a point will be called "circles at P", and euclidean circles will be designated as such. All conditions to be imposed on these circles are as follows:

[β_1] If $P \neq Q$, $Q < \overline{d}(P,X)$, then the intersection of the closed interiors of circles at Q and $d(P,X)$ is either (i) the empty set, or (ii) a single point, or (iii) a set with interior points. If $P \neq Q$, $Q < d(P,X)$, then the intersection of the closed interiors of circles at Q and $\overline{d}(P,X)$ is either (i) the empty set, or (ii) a single point, or (iii) a set with interior points.

[β_2] If for P, Q, and R, any three distinct points, any one of

(i) $c(P,Q) \cdot c(R,Q) = Q$
(ii) $c(P,R) \cdot c(Q,R) = R$
(iii) $c(R,P) \cdot c(Q,P) = P$

hold, then the other two hold.

[β_3] If $d(P,X) \cdot \overline{d}(Q,X) = X$, then $P < d(X,Q)$.

[β_4] If for $c(M,A) \neq c(M,B)$ with $A < d(M,B)$ there exist two points P and Q and circles $c(P,X)$ and

c(Q,Y) such that either (a) or (b) holds:

(a) $d(P,X) \cdot d(M,A) = \text{point}$, $d(P,X) \cdot c(M,B) = \text{point}$,

$d(Q,Y) \cdot d(M,A) = \text{point}$, $d(Q,Y) \cdot c(M,B) = \text{point}$;

(b) $\overline{d}(P,X) \cdot c(M,A) = \text{point}$, $c(P,X) \cdot c(M,B) = \text{point}$,

$\overline{d}(Q,Y) \cdot d(M,A) = \text{point}$, $c(Q,Y) \cdot c(M,B) = \text{point}$;

then c(M,P) = c(M,Q), and Q is on c(P,X) if and only if P is on c(Q,Y).

[β_5] For any three distinct points not satisfying the hypothesis of [β_2], there exists at least one circle containing these points.

The methods to be used in applying these conditions are primarily generalizations of the corresponding cases for the euclidean circles, as applied to S. L. planes.

We obtain several convergence relations for the curves of any one system $\oslash(P)$ in terms of sequences of points and their corresponding parameter values. However, the prime result is that obtained in terms of the concept of closed limit, and we are able to show:

Theorem If $\{A_\nu\}$ is a sequence of points converging to $A \neq P$, then for a given $\epsilon > 0$ there exists a number M such that:

$$d(P,A_\nu) < d(P,A) + \epsilon$$
$$d(P,A) < d(P,A_\nu) + \epsilon, \text{ for } \nu > M,$$

where $d(P,A) + \epsilon$, represents those points of d(P,A) and the open euclidean disks of radius ϵ, center on c(P,A).

This theorem, when used in conjunction with [β_1], is the

basis for the definition of the segment and so the ray and the line. We establish a one to one correspondence for those circles at two distinct points A and B such that their intersection is a single point. This is valid both for internal and external intersections, and we define the segment determined by two distinct points, as follows:

Definition Segment. If A and B are two distinct points, the segment $\overline{AB}$ is the class of points $\{ X \}$ such that

$$d(A,X) \cdot d(B,X) = X.$$

A and B are the endpoints of the segment.

The ray $\overrightarrow{AB}$ is defined simply as the extension of the segment, i.e., those points X such that B is on $\overline{AX}$. The line $\underline{AB}$ is then the composite class of points that are contained in either of the two rays or the segment $\overline{AB}$. Although these definitions of ray and line involve circles, it is only indirectly. By means of the conditions $[\beta_2]$ and $[\beta_3]$, the points on a line (and so a ray) can be determined explicitly as points of intersection of circles, concluding with the

Theorem Any point X is on $\underline{AB}$ if and only if there exist two circles $c(A,X)$ and $c(B,X)$ such that their intersection is X.

Utilizing this theorem and the one to one correspondence between the points of a line and the circles at two distinct points, we are able to verify that the straight line is uniquely determined by any of its points. The verification is accomplished by proving a series of properties similar to betweenness for the segment, ray and line.

The points of any straight line $\underline{AB}$ are determined by the circles at A and at B. Moreover, using the term *half-line* at A to mean $\overline{AB} + \overrightarrow{AB}$ or $\overline{AB} + \overrightarrow{BA}$, we can show that any circle

at C on $\underline{AB}$ determines two points of $\underline{AB}$, one on each half-line at C. Now we are able to establish a correspondence between the points of an arbitrary line and the real numbers, since the circles at any point can be parametrized, and so we have the following:

Theorem The straight line is homeomorphic to the euclidean straight line in an order preserving manner.

At this point we have a system of curves in the euclidean plane that satisfy the hypotheses of the inverse solution for the open S. L. plane. However, it will be necessary to develop a metric so that the preassigned curves will be the circles for this S. L. plane.

III. The Metric

In order to define a metric that will possess the prechosen curves as circles, and at the same time be the metric for the straight lines we use the concept of midpoint of a segment, which we define as follows:

Definition Midpoint. For any segment $\overline{AB}$, the point M of $\overline{AB}$ such that $c(M,A) = c(M,B)$ will be called the midpoint of $\overline{AB}$, and written $M = m(AB)$.

With an argument involving convergence of circles we establish that any two distinct points determine a unique midpoint. Restating $[\beta_4]$ in terms of midpoints we are then able to deduce several more obvious, but useful properties of midpoints.

With the midpoint established, we are now in a position to define a metric on a line. We do this in steps; however, in brief, utilizing the fact that an arbitrary line h is homeomorphic to the real axis, we establish a second homeomorphism of the real axis on itself in such a way that the real number midpoint and the image of the segment midpoint coincide. Selecting an arbitrary line h, we choose two points on it, one

to be the zero point, the other the unit point. Using a chain of circles we designate those points to correspond to the integral points of the real axis. We follow this, using the existence of the midpoint, to designate those points that are to have as their image rational points of the form $n/2^m$, where m and n are integers. This designation can be carried out in such a way that the numerical order of the points is consistent with the segment order. Thus we have explicitly given the correspondence for points whose images are numbers of the form $n/2^m$, and we have a preservation of order for all points. This enables us to extend the correspondence uniquely to all points of h in such a way that h is mapped on the real axis homeomorphically. If we let B_s be the point of h corresponding to the real number s and call the map F, the metric r( ) is defined to be

$$r(B_s,B_t) = | F(B_s) - F(B_t) | = | s - t | ,$$

and we can show that the one-dimensional circles of this metric are precisely the one-dimensional preassigned circles. For a different choice of the zero and unit points, the resulting metric will agree with the original metric except for a constant factor.

This metric r( ), now valid in h is then extended to the plane by a series of homeomorphisms. By means of $[\beta_4]$ we can establish a homeomorphism between h and any line in the pencil of lines at B_o that preserves the midpoints. This correspondence is completely determined by the circles at B_o, as the image of any point of h is on the same circle at B_o. Consequently, the metric r( ), defined on h can be transferred to any line through B_o. To accomplish the transfer of the metric to any line g in the plane we use an auxiliary line, determined by any point of g and B_O. Thus if P is a point of g, we first transfer r( ) to $\underline{PB_o}$ and then from $\underline{PB_o}$ to g. The validity of this extension depends on the independence of the choice of the auxiliary line, but this we establish using $[\beta_4]$ and $[\beta_5]$, showing that r( ) can be carried around any non-degenerate triangle. Thus the metric originally defined on a

chosen line h is now valid in the entire plane. For a different choice of h we find that the resulting metric differs from the original metric by a constant factor.

That r( ) is a bona fide metric for the plane, with these straight lines as geodesics, is easily verified. Moreover, from the manner in which r( ) is extended to the plane, we see that the circles of this metric are precisely the preassigned curves.

Finally, we have the space finitely compact with respect to the new metric. Since the other axioms for an open S. L. plane are trivially satisfied we conclude that our plane is an open S. L. plane.

REFERENCE

1. Busemann, H., *Metric Methods in Finsler Spaces and in the Foundations of Geometry.* Princeton: Princeton University Press, 1942.

VITA

Edgar Raymond Mullins, Jr. was born on August 16, 1923 in Champaign, Illinois. Upon graduation from Greenway High School in Coleraine, Minnesota in 1940, he attended Itasca Junior College for one year, then entered Grinnell College in September, 1941. In 1943 he enlisted in the U. S. Navy, serving in the Amphibious Forces in the Pacific Theater of Operations until his release in 1946. At that time he returned to Grinnell College and received the degree of Bachelor of Arts in 1947. In September, 1947 he entered the Graduate College of the University of Illinois. He held a half-time graduate assistantship for each academic year through 1951-1952, receiving the degree of Master of Arts in 1949. He is a member of Sigma Xi and Pi Mu Epsilon.